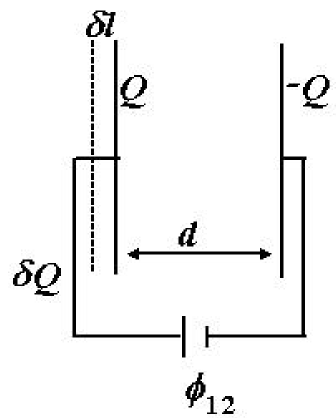
PHY 5346
HW Set 2 Solutions – Kimel

3. 1.9 I will be using the principle of virtual work. In the figure below, $F\delta l$ is the work done by an external force. If F is along δl (ie. is positive), then the force between the plates is attractive. This work goes into increasing the electrostatic energy carried by the electric field and into forcing charge into the battery holding the plates at constant potential ϕ_{12}.



Conservation of energy gives

$$F\delta l = \delta W + \delta Q\phi_{12}$$

or

$$F = \frac{\partial W}{\partial l} + \left|\frac{\partial Q}{\partial l}\right|\phi_{12}$$

From problem 1.8,

a) Charge fixed.

1) Parallel plate capacitor

$$W = \frac{1}{2}\varepsilon_0 A\frac{\phi_{12}^2}{d},\ \phi_{12} = \frac{dQ}{A\varepsilon_0} \to W = \frac{1}{2}\varepsilon_0 A\frac{\left(\frac{dQ}{A\varepsilon_0}\right)^2}{d} = \frac{d}{2\varepsilon_0 A}Q^2$$

$$\frac{\partial Q}{\partial l} = 0,\ F = \frac{\partial W}{\partial l} = \frac{Q^2}{2\varepsilon_0 A}\ \text{(attractive)}$$

2) Parallel cylinder capacitor

$$\phi_{12} = \frac{\lambda}{\varepsilon_0}\ln\!\left(\frac{d}{a}\right),\ a = \sqrt{a_1 a_2}$$

$$W = \frac{1}{2}Q\phi_{12} \to F = \frac{\partial W}{\partial l} = \frac{1}{2}Q\frac{\lambda}{\varepsilon_0}\frac{\partial}{\partial d}\ln\!\left(\frac{d}{a}\right) = \frac{1}{2}\frac{\lambda Q}{\varepsilon_0 d}\ \text{(attractive)}$$

b) Potential fixed

1) Parallel plate capacitor

$$\text{Using Gauss's law, } Q = \frac{\phi_{12}A\varepsilon_0}{d},\ \left|\frac{\partial}{\partial l}Q\right| = \frac{\phi_{12}A\varepsilon_0}{d^2}$$

$$\to F = -\frac{1}{2}\varepsilon_0 A\frac{\phi_{12}^2}{d^2} + \frac{\phi_{12}^2 A\varepsilon_0}{d^2} = \frac{1}{2}\varepsilon_0 A\frac{\phi_{12}^2}{d^2} = \frac{1}{2}\varepsilon_0 A\frac{\left(\frac{Qd}{\varepsilon_0 A}\right)^2}{d^2} = \frac{1}{2\varepsilon_0 A}Q^2$$

2) Parallel cylinder capacitor

$$W = \frac{1}{2}Q\phi_{12},\ \text{and } Q = \frac{\varepsilon_0 L\phi_{12}}{\ln(\frac{d}{a})}$$

so

$$W = \frac{1}{2}\frac{\varepsilon_0 L\phi_{12}^2}{\ln(\frac{d}{a})},\ \frac{\partial W}{\partial l} = -\frac{1}{2}\varepsilon_0 L\frac{\phi_{12}^2}{\left(\ln^2\frac{d}{a}\right)d}$$

$$\left|\frac{\partial Q}{\partial l}\right| = \varepsilon_0 L\frac{\phi_{12}}{\left(\ln^2\frac{d}{a}\right)d}$$

$$F = -\frac{1}{2}\varepsilon_0 L\frac{\phi_{12}^2}{\left(\ln^2\frac{d}{a}\right)d} + \varepsilon_0 L\frac{\phi_{12}^2}{\left(\ln^2\frac{d}{a}\right)d} = \frac{1}{2}\varepsilon_0 L\frac{\phi_{12}^2}{\left(\ln^2\frac{d}{a}\right)d}$$