PHY 5346
HW Set 7 Solutions – Kimel

3. 4.7 a) Since ρ does not depend on ϕ, we can write it in terms of spherical harmonics with $m = 0$. First note

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\left(1-\sin^2\theta\right) - \frac{1}{2}\right)$$

or

$$\sin^2\theta = -\frac{2}{3}\sqrt{\frac{4\pi}{5}}\,Y_2^0 + \sqrt{4\pi}\,\frac{2}{3}Y_0^0$$

Thus only the $m = 0, l = 0, 2$ multipoles contribute.

$$q_{00} = \frac{2\sqrt{4\pi}}{3}\int_0^\infty r^2\left(\frac{1}{64\pi}r^2e^{-r}\right)dr = \frac{2\sqrt{4\pi}}{3}\frac{3}{8\pi} = \frac{1}{2\sqrt{\pi}}$$

$$q_{20} = -\frac{2}{3}\sqrt{\frac{4\pi}{5}}\int_0^\infty r^4\left(\frac{1}{64\pi}r^2e^{-r}\right)dr = -\frac{2}{3}\sqrt{\frac{4\pi}{5}}\frac{45}{4\pi} = -3\frac{\sqrt{5}}{\sqrt{\pi}}$$

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\left[4\pi q_{00}\frac{Y_0^0}{r} + 4\pi q_{20}\frac{Y_2^0}{5r^3}\right] = \frac{1}{4\pi\varepsilon_0}\left[\sqrt{4\pi}\,q_{00}\frac{P_0}{r} + \sqrt{\frac{4\pi}{5}}\,q_{20}\frac{P_2}{r^3}\right]$$

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{P_0}{r} - 6\frac{P_2}{r^3}\right]$$

b)

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\vec{x}')d^3x'}{|\vec{x}-\vec{x}'|}$$

Using

$$\frac{1}{|\vec{x}-\vec{x}'|} = 4\pi\sum_{lm}\frac{1}{(2l+1)r_>}\left(\frac{r_<}{r_>}\right)^l Y_l^{m*}(\theta',\phi')Y_l^m(\theta,\phi)$$

, we see only the $l = 0, 2$ and $m = 0$ terms of the expansion contribute in the potential. Next take $r' > r$.

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}4\pi\sum_{lm}\frac{1}{(2l+1)}r^lY_l^m(\theta,\phi)\int Y_l^{m*}(\theta',\phi')r'^2d\Omega'\frac{\rho(\vec{x}')}{r'^{l+1}}dr'$$

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}4\pi\left[Y_0^0\sqrt{4\pi}\,\frac{2}{3}\int_0^\infty\left(\frac{1}{64\pi}r^2e^{-r}\right)rdr + \frac{Y_2^0}{5}r^2\left(-\frac{2}{3}\sqrt{\frac{4\pi}{5}}\int_0^\infty\left(\frac{1}{64\pi}r^2e^{-r}\right)\frac{1}{r}dr\right)\right]$$

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}4\pi\left[Y_0^0\sqrt{4\pi}\,\frac{2}{3}\frac{3}{32\pi} + \frac{Y_2^0}{5}r^2\left(-\frac{2}{3}\sqrt{\frac{4\pi}{5}}\right)\frac{1}{64\pi}\right]$$

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}4\pi\left[P_0\frac{2}{3}\frac{3}{32\pi} + \frac{P_2}{5}r^2\left(-\frac{2}{3}\right)\frac{1}{64\pi}\right] = \frac{1}{4\pi\varepsilon_0}\left[\frac{P_0}{4} - \frac{r^2P_2}{120}\right]$$